Journal of Applied Mechanics and Technical Physics, Vol. 51, No. 2, pp. 269–279, 2010

MODELING OF PARTIAL CLOSURE OF CRACKS IN A PERFORATED ISOTROPIC MEDIUM REINFORCED BY A REGULAR SYSTEM OF STRINGERS

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UDC 539.375

A mathematical model of partial closure of a crack in a perforated isotropic medium with a system of rectilinear foreign inclusions is constructed. Such a medium can be interpreted as an unbounded plate reinforced by a regular system of ribs whose cross sections are shaped as narrow rectangles. The medium is assumed to be attenuated by a periodic system of circular holes and straight-line cracks. Determination of unknown contact stresses and contact zone sizes is reduced to solving a singular integral equation, which is transformed by an algebraization procedure to a system of nonlinear algebraic equations solved by the method of consecutive approximations.

Key words: *perforated reinforced plate, stringers, cracks, contact zone, contact stresses.*

One of the effective methods of delaying the crack growth is the use of reinforcing stiffness elements on the way of crack propagation [1, 2]. The problem of "healing" of the existing crack is extremely important in the fracture theory. The first stage of solving this problem is the problem of closure of an open crack.

1. Formulation of the Problem. The results obtained in [1–3] show that the use of reinforcing elements makes it possible to reduce deformation of the stretched plate in the direction perpendicular to crack propagation and, hence, the stress intensity factor in the vicinity of the crack tip. Certain combinations of physical and geometrical parameters of the reinforced plate give rise to zones with compressive stresses, where the crack faces interact on a certain segment, which leads to generation of contact stresses.

Let us consider an elastic isotropic medium with a periodic system of circular holes of radius λ whose contours are free from external forces. Transverse stiffness ribs made of another elastic material and having a cross-sectional area A_s are riveted to the plate.

The chosen system of Cartesian coordinates and the notations used are shown in Fig. 1. The reinforced plate is subjected at infinity to uniform tension along the stringers with the stress $\sigma_y^\infty = \sigma_0$.

The hypothesis assumed for the stringer implies that the stringer thickness remains unchanged during its deformation, and the stress state is uniaxial. The stringers are not subjected to bending and experience tensile forces only.

The following assumptions are made:

1) a plane stress state is formed in the thin-walled sheet element of the structure (plate);

2) attenuation of the stringers due to attachment points is ignored;

3) the sheet element and the reinforcing elements interact in one plane and only at attachment points;

4) all attachment points are identical, and their radius (attachment area) is small, as compared with the distance between the attachment points and with other characteristic sizes;

5) interaction between the stringer and the plate is modeled by point forces (see Fig. 1).

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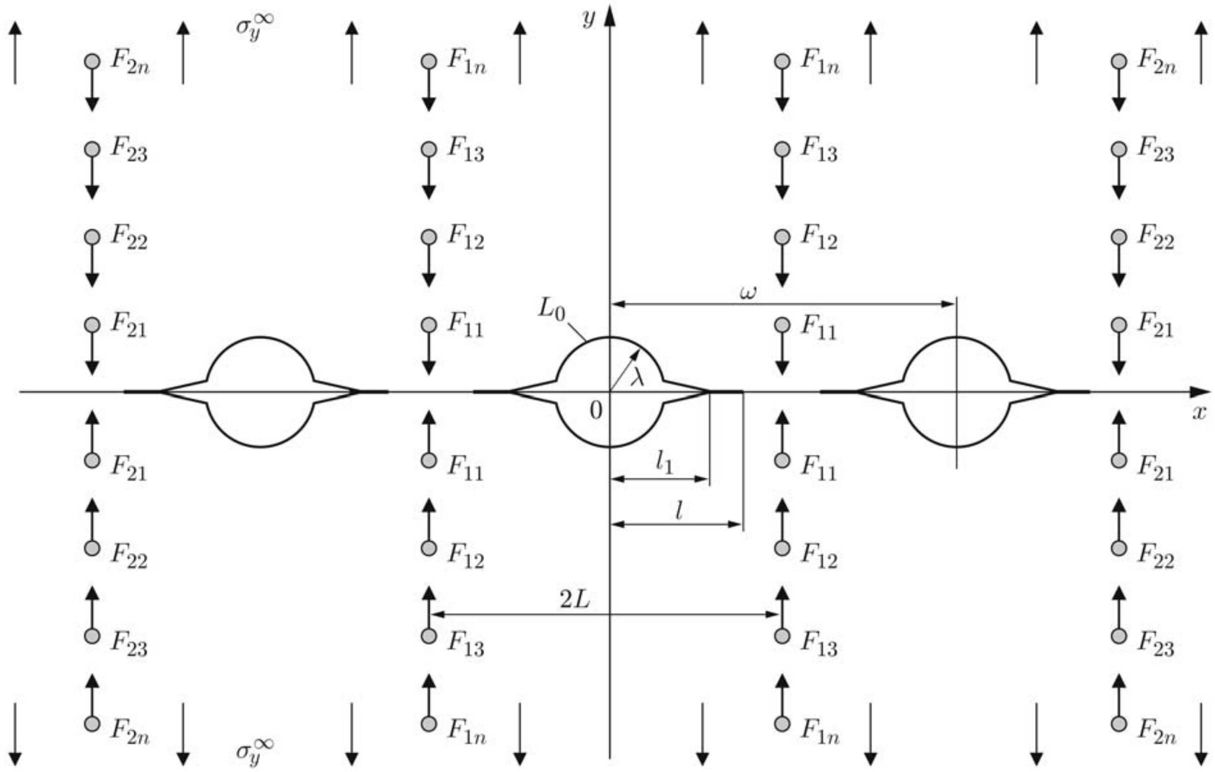

Fig. 1. Computational scheme of the problem.

The attachment points of the stringers are assumed to be aligned over the entire length of the stringer at an identical distance from each other and symmetrically with respect to the plate surface (see Fig. 1).

The action of the stringers is modeled in the calculation scheme by unknown equivalent point forces applied at the points where the stringers are connected to the plate. As the intensity of external loading is increased, zones with elevated stresses are formed in the reinforced plate; these zones are located periodically. The presence of elevated stress zones favors the emergence of surface cracks.

Let us consider a reinforced isotropic medium with a periodic system of circular holes of radius λ ($\lambda < 1$) with the hole centers being located at the points

$$P_m = m\omega, \qquad m = \pm 1, \pm 2, \ldots, \qquad \omega = 2.$$

Symmetric straight-line cracks emanate from the hole contours (see Fig. 1). The crack faces are free from external loading. Because of the symmetry of the boundary conditions and the geometry of the domain D occupied by the medium, the stresses are periodic functions with a period ω.

Under the action of an external tensile load σ_0 and point forces F_{mn} ($m = \pm 1, \pm 2, \ldots$, $n = \pm 1, \pm 2, \ldots$) to be determined in the course of solving the problem, the crack faces interact on certain segments of the zone of compressive stresses, which leads to the emergence of contact stresses on the segments considered. The crack portions outside these zones are free from stresses.

The parameters l_1 and l of the boundary of the contact zone of the crack faces have to be determined in the course of solving the problem. It should be noted that the zone of interaction of the crack faces in the problem considered always begins at the crack tip located in the zone of compressive stresses.

Our task is to develop a mathematical model for finding the contact stresses on the segments $(-l+m\omega, -l_1+m\omega)$ and $(l_1+m\omega, l+m\omega)$, the values of the point forces F_{mn} ($m = \pm 1, \pm 2, \ldots$; $n = \pm 1, \pm 2, \ldots$), and the stress–strain state outside the circular holes and cracks.

Let us consider segments of length $l_0 = l - l_1$ (tip contact zones) adjacent to the crack tip, where the crack faces interact. It should be noted that interaction of the crack faces prevents crack opening.

Normal stresses $p(x)$ arise in the tip zones where the crack faces are in contact. The values of these contact stresses and the contact zone size are not known in advance and have to be determined in the course of solving the problem. In the case considered, the surface of each crack has two zones: the tip contact zone and the zone where the crack faces are free from loading.

The boundary conditions on the crack faces have the form

$$\sigma_y = 0, \qquad \tau_{xy} = 0 \qquad \text{on} \quad L',$$

$$\sigma_y = p(x), \qquad \tau_{xy} = 0, \qquad v^+(x,0) - v^-(x,0) = 0 \qquad \text{on} \quad L''.$$

Here, L' is the set of zones on the crack surface free from loading, L'' is the set of tip zones where the crack faces interact, $v^+(x,0) - v^-(x,0)$ is the opening of the crack faces, σ_x, σ_y, and τ_{xy} are the stress tensor components, and u and v are the components of the displacement vector in the x and y directions, respectively.

The boundary conditions of the problem on the circular hole contours are

$$\sigma_r - i\tau_{r\theta} = 0.$$

On the basis of the formulas [4]

$$\sigma_x + \sigma_y = \sigma_r + \sigma_\theta = 2[\Phi(z) + \overline{\Phi(z)}\,], \qquad z = x + iy, \quad \sigma_y - \sigma_x + 2i\tau_{xy} = (\sigma_\theta - \sigma_r + 2i\tau_{r\theta})\, e^{-2i\theta} = 2[\bar{z}\Phi'(z) + \Psi(z)]$$

and the boundary conditions on the circular hole contours and on the crack faces, the problem solution is reduced to determining two functions $\Phi(z)$ and $\Psi(z)$ analytical in the domain D from the boundary conditions

$$\Phi(\tau) + \overline{\Phi(\tau)} - [\bar{\tau}\Phi'(\tau) + \Psi(\tau)]\, e^{2i\theta} = 0; \tag{1.1}$$

$$\Phi(x) + \overline{\Phi(x)} + x\,\overline{\Phi'(x)} + \overline{\Psi(x)} = f, \tag{1.2}$$

where $\tau = \lambda\, e^{i\theta} + m\omega$ $(m = 0, \pm 1, \pm 2, \ldots)$, x is the affix of the crack face points, and $f = 0$ on L' and $f = p(x)$ on L''.

2. Solution of the Boundary-Value Problem. The solution of the boundary-value problem (1.1), (1.2) is sought in the form

$$\Phi(z) = \Phi_0(z) + \Phi_1(z) + \Phi_2(z), \qquad \Psi(z) = \Psi_0(z) + \Psi_1(z) + \Psi_2(z). \tag{2.1}$$

Here, the complex potentials $\Phi_0(z)$ and $\Psi_0(z)$ determine the stress and strain fields, respectively, in the reinforced solid plate under the action of the tensile stress σ_0 and the point forces F_{mn}:

$$\begin{gathered} \Phi_0(z) = \frac{1}{4}\,\sigma_0 - \frac{i}{2\pi h(1+\varkappa_0)} \sum_{m,n}{}' F_{mn}\Big(\frac{1}{C_1} - \frac{1}{C_2}\Big), \\ \Psi_0(z) = \frac{1}{2}\,\sigma_0 - \frac{i\varkappa_0}{2\pi h(1+\varkappa_0)} \sum_{m,n}{}' F_{mn}\Big(\frac{1}{C_1} - \frac{1}{C_2}\Big) + \frac{i}{2\pi h(1+\varkappa_0)} \sum_{m,n}{}' F_{mn}\Big(\frac{\bar{C}_3}{C_2^2} - \frac{C_3}{C_1^2}\Big). \end{gathered} \tag{2.2}$$

Here, $C_1 = z - mL + iny_0$, $C_2 = z - mL - iny_0$, and $C_3 = mL + iny_0$, h is the plate thickness, $\varkappa_0 = (3-\nu)/(1+\nu)$, and ν is Poisson's ratio of the plate material; the prime at the summation sign indicates that the subscript $m = n = 0$ is eliminated in summation.

The functions $\Phi_1(z)$ and $\Psi_1(z)$ corresponding to unknown normal displacements along the crack are sought in the explicit form as

$$\begin{gathered} \Phi_1(z) = \frac{1}{2\omega} \int\limits_{L_1} g(t) \cot \frac{\pi}{\omega}\,(t - z)\, dt, \\ \Psi_1(z) = -\frac{\pi z}{2\omega^2} \int\limits_{L_1} g(t) \sin^{-2} \frac{\pi}{\omega}\,(t - z)\, dt, \qquad L_1 = [-l, -\lambda] + [\lambda, l]. \end{gathered} \tag{2.3}$$

Here the sought function $g(x)$ characterizes the derivative of opening of the crack faces:

$$\frac{1+\varkappa_0}{2\mu}\, g(x) = \frac{\partial}{\partial x}\, [v^+(x,0) - v^-(x,0)]$$

(μ is the shear modulus of the reinforced plate).

To find the complex potentials $\Phi_2(z)$ and $\Psi_2(z)$, we present the boundary condition (1.1) in the form

$$\Phi_2(\tau) + \overline{\Phi_2(\tau)} - [\bar{\tau}\Phi_2'(\tau) + \Psi_2(\tau)]\,\mathrm{e}^{2i\theta} = f_1(\theta) + if_2(\theta) + \varphi_1(\theta) + i\varphi_2(\theta), \tag{2.4}$$

where

$$f_1(\theta) + if_2(\theta) = -\Phi_0(\tau) - \overline{\Phi_0(\tau)} + [\bar{\tau}\Phi_0'(\tau) + \Psi_0(\tau)]\,\mathrm{e}^{2i\theta}; \tag{2.5}$$

$$\varphi_1(\theta) + i\varphi_2(\theta) = -\Phi_1(\tau) - \overline{\Phi_1(\tau)} + [\bar{\tau}\Phi_1'(\tau) + \Psi_1(\tau)]\,\mathrm{e}^{2i\theta}\,. \tag{2.6}$$

The complex potentials $\Phi_2(z)$ and $\Psi_2(z)$ are sought in the form

$$\begin{gathered}\Phi_2(z) = \alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2}\,\frac{\lambda^{2k+2}\rho^{(2k)}(z)}{(2k+1)!},\\ \Psi_2(z) = \sum_{k=0}^{\infty} \beta_{2k+2}\,\frac{\lambda^{2k+2}\rho^{(2k)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} \alpha_{2k+2}\,\frac{\lambda^{2k+2}S^{(2k+1)}(z)}{(2k+1)!},\end{gathered} \tag{2.7}$$

where

$$\rho(z) = \Big(\frac{\pi}{\omega}\Big)^2 \sin^{-2}\Big(\frac{\pi}{\omega}\,z\Big) - \frac{1}{3}\Big(\frac{\pi}{\omega}\Big)^2, \qquad S(z) = {\sum_{m,n}}' \Big(\frac{P_m}{(z-P_m)^2} - \frac{2z}{P_m^2} - \frac{1}{P_m}\Big).$$

The conditions of symmetry about the coordinate axes yield the equalities

$$\operatorname{Im}\alpha_{2k+2} = 0, \qquad \operatorname{Im}\beta_{2k+2} = 0, \qquad k = 0, 1, 2, \ldots.$$

Relations (2.1)–(2.3) and (2.7) determine a class of symmetric problems with a periodic distribution of stresses.

The condition of a constant principal vector of forces acting on the arc connecting two congruent points in the domain D implies that

$$\alpha_0 = \pi^2\beta_2\lambda^2/24.$$

The unknown coefficients α_{2k+2} and β_{2k+2} have to be determined from the boundary condition (2.4). We assume that the functions $f_1(\theta) + if_2(\theta)$ and $\varphi_1(\theta) + i\varphi_2(\theta)$ are decomposed into a Fourier series on the contour $|\tau| = \lambda$. By virtue of symmetry, these series have the form

$$\begin{gathered}f_1(\theta) + if_2(\theta) = \sum_{k=-\infty}^{\infty} A_{2k}\,\mathrm{e}^{2ik\theta}, \qquad \operatorname{Im}A_{2k} = 0,\\ A_{2k} = \frac{1}{2\pi}\int_0^{2\pi} (f_1(\theta) + if_2(\theta))\,\mathrm{e}^{-2ik\theta}\,d\theta \qquad (k = 0, \pm1, \pm2, \ldots);\end{gathered} \tag{2.8}$$

$$\begin{gathered}\varphi_1(\theta) + i\varphi_2(\theta) = \sum_{k=-\infty}^{\infty} B_{2k}\,\mathrm{e}^{2ik\theta}, \qquad \operatorname{Im}B_{2k} = 0,\\ B_{2k} = \frac{1}{2\pi}\int_0^{2\pi} (\varphi_1(\theta) + i\varphi_2(\theta))\,\mathrm{e}^{-2ik\theta}\,d\theta \qquad (k = 0, \pm1, \pm2, \ldots).\end{gathered} \tag{2.9}$$

Substituting Eq. (2.5) into Eq. (2.8) and calculating the integrals with the use of the theory of residues, we obtain

$$A_0 = -\frac{1}{2}\,\sigma_0 + \frac{1}{\pi h(1+\varkappa_0)} {\sum_{m,n}}' F_{mn}\,\frac{2ny_0}{C_3\bar{C}_3},$$

$$A_2 = \frac{1}{2}\,\sigma_0 - \frac{1}{\pi h(1+\varkappa_0)} {\sum_{m,n}}' F_{mn}\Big(\frac{\lambda^2\sin 3\varphi_3}{\rho_1^3} + \frac{\varkappa_0\sin\varphi_3}{\rho_1} - \frac{\sin 3\varphi_3}{\rho_1}\Big),$$

$$A_{2k} = \frac{1}{\pi h(1+\varkappa_0)}\Big[\sum_{m,n}{}' F_{mn}\Big(\frac{\lambda^{2k}\sin{(2k+1)\varphi_3}}{\rho_1^{2k+1}} + \frac{(-2)(-3)\cdots(-2k)\lambda^{2k}\sin{(2k+1)\varphi_3}}{(2k-1)!\rho_1^{2k+1}}$$

$$- \frac{\varkappa_0\lambda^{2k-2}\sin{(2k-1)\varphi_3}}{\rho_1^{2k-1}} + \frac{(-2)(-3)\cdots(1-2k)\lambda^{2k-2}\sin{(2k+1)\varphi_3}}{(2k-1)!\rho_1^{2k-1}}\Big)\Big] \qquad (k = 2, 3, \ldots),$$

$$A_{-2k} = \frac{1}{\pi h(1+\varkappa_0)}\sum_{m,n}{}' F_{mn}\,\frac{\lambda^{2k}\sin{(2k+1)\varphi_3}}{\rho_1^{2k+1}} \qquad (k = 1, 2, \ldots),$$

$$\rho_1 = \sqrt{C_3\bar{C}_3}, \qquad \varphi_3 = \arctan\frac{ny_0}{mL}.$$

Similarly, substituting Eq. (26) into Eq. (2.9) and calculating the integrals with the use of the theory of residues, we obtain

$$B_{2k} = -\frac{1}{2\omega}\int\limits_{L_1} g(t)f_{2k}(t)\,dt,$$

where

$$f_0(t) = 2\gamma(t), \qquad f_2(t) = -\frac{\lambda^2}{2}\,\gamma^{(2)}(t), \qquad \gamma(t) = \cot\frac{\pi}{\omega}\,t,$$

$$f_{2k}(t) = -\frac{\lambda^{2k}(2k-1)}{(2k)!}\,\gamma^{(2k)}(t) + \frac{\lambda^{2k-2}}{(2k-3)!}\,\gamma^{(2k-2)}(t) \qquad (k = 2, 3, \ldots),$$

$$f_{-2k}(t) = -\frac{\lambda^{2k}}{(2k)!}\,\gamma^{(2k)}(t) \qquad (k = 1, 2, \ldots).$$

The unknown function $g(x)$ and the coefficients α_{2k} and β_{2k} are determined from the boundary conditions (1.2) and (2.4). As the periodicity conditions are satisfied, the system of the boundary conditions (2.4) degenerates into one functional equation, for instance, on the contour L_0 ($\tau = \lambda\,e^{i\theta}$), and the system of the boundary conditions (1.2) degenerates into the boundary condition on the line L_1.

To construct equations with respect to the coefficients α_{2k} and β_{2k} of the functions $\Phi_2(z)$ and $\Psi_2(z)$, we expand these functions into the Laurent series in the neighborhood of the point $z = 0$. Substituting, instead of $\Phi_2(z)$, $\overline{\Phi_2(z)}$, $\Phi_2'(z)$, and $\Psi_2(z)$, their expansions into the Laurent series in the neighborhood of $z = 0$ into the left side of the boundary condition (2.4) on the contour $z = \lambda\exp{(i\theta)}$, substituting the Fourier series (2.8), (2.9) instead of the functions $f_1(\theta) + if_2(\theta)$ and $\varphi_1(\theta) + i\varphi_2(\theta)$ into the right side of Eq. (2.4), and comparing the coefficients at identical powers of $\exp{(i\theta)}$, we obtain two infinite systems of algebraic equations with respect to the coefficients α_{2k} and β_{2k}. After some transformations, we obtain an infinite system of linear algebraic equations with respect to α_{2k+2}:

$$\alpha_{2j+2} = \sum_{k=0}^{\infty} A_{j,k}\alpha_{2k+2} + b_j \qquad (j = 0, 1, 2, \ldots). \tag{2.10}$$

In this system, we have

$$b_0 = M_2 - \sum_{k=0}^{\infty}\frac{g_{k+2}\lambda^{2k+4}}{2^{2k+4}}\,M_{-2k-2},$$

$$b_j = M_{2j+2} - \frac{(2j+1)M_0 g_{j+1}\lambda^{2j+2}}{K_1 2^{2j+2}} - \sum_{k=0}^{\infty}\frac{(2j+2k+3)g_{j+k+2}\lambda^{2j+2k+4}}{(2j)!(2k+3)!2^{2j+2k+4}}\,M_{-2k-2},$$

$$A_{j,k} = (2j+1)\gamma_{j,k}\lambda^{2j+2k+2}, \qquad K_1 = 1 - \frac{\pi^2}{12}\,\lambda^2, \qquad g_j = 2\sum_{m=1}^{\infty}\frac{1}{m^{2j}},$$

$$\gamma_{0,0} = \frac{3}{8} g_2 \lambda^2 + \sum_{i=1}^{\infty} \frac{(2i+1) g_{i+1}^2 \lambda^{4i+2}}{2^{4i+4}},$$

$$\gamma_{j,k} = -\frac{(2j+2k+2)! g_{k+j+1}}{(2j+1)!(2k+1)! 2^{2j+2k+2}} + \frac{(2j+2k+4)! g_{j+k+2} \lambda^2}{(2j+2)!(2k+2)! 2^{2j+2k+4}}$$

$$+ \sum_{i=0}^{\infty} \frac{(2j+2i+1)!(2k+2i+1)! g_{j+i+1} g_{k+i+1} \lambda^{4i+2}}{(2j+1)!(2k+1)!(2i+1)!(2i)! 2^{2j+2k+4i+4}} + b_{j,k},$$

$$b_{0,k} = 0, \quad b_{j,0} = 0, \quad b_{j,k} = \frac{g_{j+1} g_{k+1} \lambda^2}{2^{2j+2k+4}} \Big(1 + \frac{2K_2 \lambda^2}{K_1}\Big) \quad (j = 1, 2, \ldots, \; k = 1, 2, \ldots),$$

$$K_2 = \pi^2/24, \qquad M_{2k} = A_{2k} + B_{2k}.$$

The constants β_{2k+2} are determined from the following relations:

$$\beta_2 = \frac{1}{K_1} \Big(-M_0 + 2 \sum_{k=0}^{\infty} \frac{g_{k+1} \lambda^{2k+2}}{2^{2k+2}} \alpha_{2k+2} \Big),$$

$$\beta_{2j+4} = (2j+3) \alpha_{2j+2} + \sum_{k=0}^{\infty} \frac{(2j+2k+3)! g_{j+k+2} \lambda^{2j+2k+4}}{(2j+2)!(2k+1)! 2^{2j+2k+4}} \alpha_{2k+2} - M_{-2j-2}. \tag{2.11}$$

Requiring that functions (2.1) satisfy the boundary condition (1.2) and applying some transformations, we obtain a singular integral equation with respect to the function $g(x)$:

$$\frac{1}{\omega} \int_{L_1} g(t) \cot \frac{\pi}{\omega} (t - x) \, dt + H(x) = f(x). \tag{2.12}$$

Here, $H(x) = \Phi_*(x) + \overline{\Phi_*(x)} + x\Phi_*'(x) + \Psi_*(x)$, $\Phi_*(x) = \Phi_0(x) + \Phi_2(x)$, and $\Psi_*(x) = \Psi_0(x) + \Psi_2(x)$.

The singular integral equation (2.12) and also the algebraic systems (2.10), (2.11) contain unknown point forces F_{mn} ($m = 1, 2, \ldots$; $n = 1, 2, \ldots$). To find the value of the point force F_{mn}, we use Hooke's law and the method of "matching" of two asymptotic curves of the sought solutions. According to Hooke's law, the value of the point force F_{mn} acting onto each point of attachment from the side of the stringer is

$$F_{mn} = \frac{E_s A_s}{2y_0 n} \Delta v_{m,n} \qquad (m = 1, 2, \ldots, \; n = 1, 2, \ldots).$$

Here E_s is Young's modulus of the stringer material, $2y_0 n$ is the distance between the attachment points, and $\Delta v_{m,n}$ is the dimensionless displacement of the considered attachment points, which is equal to the elongation of the corresponding segment of the stringer.

Let us denote the radius of the attachment points (attachment area) by a_0. We use a natural assumption that the relative elastic displacement of the points $z = mL + i(ny_0 - a_0)$ and $z = mL - i(ny_0 - a_0)$ in the considered problem of the elasticity theory equals the relative displacement of the attachment points $\Delta v_{m,n}$. This additional condition of compatibility of displacements allows us to find the solution of the problem posed above.

Using the complex potentials (2.1)–(2.3) and (2.7) and the Kolosov–Muskhelishvili formulas for displacements [4], we find the dimensionless displacement $\Delta v_{m,n}$:

$$\Delta v_{p,r} = \Delta v_{p,r}^{(0)} + \Delta v_{p,r}^{(1)} + \Delta v_{p,r}^{(2)}. \tag{2.13}$$

Here,

$$\Delta v_{p,r}^{(0)} = \frac{1}{2\pi h \mu (1 + \varkappa_0)} \sum_{m,n}{}' F_{mn} \Bigg(\varkappa_0 \ln \frac{(p-m)^2 L^2 + a_0^2}{(p-m)^2 L^2 + C^2}$$

$$+ \frac{2(r-n) y_0 C [2p(p-m)L^2 + a_0 C]}{[(p-m)^2 L^2 + C^2][(p-m)^2 L^2 + a_0^2]} \Bigg) + \frac{\sigma_0}{4\mu} (1 + \varkappa_0)(r y_0 - a_0),$$

$$\Delta v_{p,r}^{(1)} = \frac{1+\varkappa_0}{\mu}\left\{\frac{1}{2\omega}\int_{L_1} g(t)\Big[\arctan\Big(\cot\frac{\pi}{\omega}(t-pL)\tanh\frac{\pi}{\omega}C\Big) - \arctan\Big(\tan\frac{\pi pL}{\omega}\tanh\frac{\pi}{\omega}C\Big)\Big]\,dt\right\}$$

$$-\frac{C}{\mu}\frac{1}{2\omega}\int_{L_1} g(t)\Big(\frac{\sin^2\alpha_1(\cosh^2\alpha_1+\sinh^2\alpha_1)}{\sin^2\alpha_1\cosh^2\alpha_1+\cos^2\alpha_1\sinh^2\alpha_1}\Big)\,dt,$$

$$\Delta v_{p,r}^{(2)} = \frac{1}{\mu}\Bigg((\varkappa_0-1)(ry_0-a_0)a_0 + (1+\varkappa_0)\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}\sin(2k+1)\alpha}{(2k+1)\rho_2^{2k+1}}$$

$$+(\varkappa_0-1)\sum_{k=0}^{\infty}\alpha_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}\frac{r_{j,k}}{2j+1}\rho_2^{2j+1}\sin(2j+1)\alpha$$

$$-\sum_{k=0}^{\infty}\beta_{2k+2}\frac{\lambda^{2k+2}\sin(2k+1)\alpha}{(2k+1)\rho_2^{2k+1}} - \sum_{k=0}^{\infty}\beta_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}\frac{r_{j,k}}{2j+1}\rho_2^{2j+1}\sin(2j+1)\alpha$$

$$+\sum_{k=0}^{\infty}(2k+2)\alpha_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}\frac{2j+2k+2}{2j+1}r_{j,k}\rho_2^{2j+1}\sin(2j+1)\alpha\Bigg),$$

$$C=(r-n)y_0-a_0, \qquad \alpha_1=\frac{\pi}{\omega}(t-pL), \qquad \alpha=\arctan\frac{ry_0-a_0}{pL},$$

$$\rho_2^2=(pL)^2+(ry_0-a_0)^2, \qquad r_{j,k}=\frac{(2j+2k+1)!g_{j+k+1}}{(2j)!(2k+1)!2^{2j+2k+2}}, \qquad r_{0,0}=0.$$

The sought value of the force F_{mn} is determined by Eqs. (2.13) from the infinite system

$$F_{pr}=\frac{E_sA_s}{2y_0r}\Delta v_{p,r} \qquad (p=1,2,\ldots,\ r=1,2,\ldots), \tag{2.14}$$

which degenerates into one infinite algebraic system by virtue of problem periodicity.

Thus, Eq. (2.14), the algebraic systems (2.10) and (2.11), and the singular integral equation (2.12) are interrelated and have to be solved together. Solving these equations and taking into account that there is no opening of the crack faces in the tip zone and that the contact stresses are bounded, we find the sought function $p(x)$ and the force F_{mn} and determine the contact zone of the crack faces.

3. Numerical Solution of the Problem and Analysis of Results. Using the expansion

$$\frac{\pi}{\omega}\cot\frac{\pi}{\omega}z=\frac{1}{z}-\sum_{j=0}^{\infty}g_{j+1}\frac{z^{2j+1}}{\omega^{2j+2}},$$

we can convert Eq. (2.12) to the standard form

$$\frac{1}{\pi}\int_{L_1}\frac{g(t)\,dt}{t-x}+\frac{1}{\pi}\int_{L_1}g(t)K(t-x)\,dt+H(x)=f(x), \tag{3.1}$$

where

$$K(t)=-\sum_{j=0}^{\infty}g_{j+1}\frac{t^{2j+1}}{\omega^{2j+2}}.$$

Taking into account that $g(x)$ is an odd function, we convert the integral equation (3.1) to a form more convenient for finding the approximate solution:

$$\frac{2}{\pi}\int_{\lambda_1}^{1}\frac{\xi g(\xi)\,d\xi}{\xi^2-\xi_0^2}+\frac{1}{\pi}\int_{\lambda_1}^{1}K_0(\xi,\xi_0)g(\xi)\,d\xi+H(\xi_0)=f(\xi_0). \tag{3.2}$$

Here,

$$K_0(\xi,\xi_0) = K(\xi-\xi_0) + K(\xi+\xi_0), \qquad g(t) = p(\xi), \qquad \xi = \frac{t}{l}, \qquad \xi_0 = \frac{x}{l}, \qquad \lambda_1 = \frac{\lambda}{l},$$

$$H(\xi_0) = \Phi_*(\xi_0 l) + \overline{\Phi_*(\xi_0 l)} + (\xi_0 l)\Phi'_*(\xi_0 l) + \Psi_*(\xi_0 l).$$

We replace the variables as

$$\xi^2 = u = \frac{1-\lambda_1^2}{2}(\tau+1) + \lambda_1^2, \qquad \xi_0^2 = u_0 = \frac{1-\lambda_1^2}{2}(\eta+1) + \lambda_1^2.$$

The interval of integration $[\lambda_1, 1]$ transforms to the interval $[-1, 1]$, and the transformed Eq. (3.2) acquires the standard form

$$\frac{1}{\pi}\int_{-1}^{1} \frac{g_*(\tau)}{\tau-\eta}\,d\tau + \frac{1}{\pi}\int_{-1}^{1} g_*(\tau)B(\eta,\tau)\,d\tau + H_*(\eta) = f_*(\eta), \tag{3.3}$$

where

$$g_*(\tau) = g(\xi), \qquad H_*(\eta) = H(\xi_0), \qquad f_*(\eta) = f(\xi_0),$$

$$B(\eta,\tau) = -\frac{1-\lambda_1^2}{2}\sum_{j=0}^{\infty} g_{j+1}\Big(\frac{l}{2}\Big)^{2j+2} u_0^j A_j^*,$$

$$A_j^* = 2j+1 + \frac{(2j+1)2j(2j-1)}{1\cdot 2\cdot 3}\,\frac{u}{u_0}\cdots + \frac{(2j+1)2j(2j-1)\cdots[2j+1-(2j+1-1)]}{1\cdot 2\cdot 3\cdots(2j+1)}\Big(\frac{u}{u_0}\Big)^j.$$

The solution of the singular integral equation (3.3) is presented in the form [5, 6]

$$g_*(\eta) = g_0(\eta)/\sqrt{1-\eta^2}. \tag{3.4}$$

Here, the function $g_0(\eta)$ is continuous according to Hölder on the interval $[-1, 1]$. The function $g_0(\eta)$ is replaced by the interpolation Lagrange polynomial constructed over the Chebyshev nodes:

$$L_n[g_0;\eta] = \frac{1}{M}\sum_{k=1}^{M}(-1)^{k+1} g_k^0 \frac{\cos M\theta\,\sin\theta_k}{\cos\theta - \cos\theta_k}. \tag{3.5}$$

Here,

$$g_k^0 = g_*(\eta_k), \qquad \eta = \cos\theta, \qquad \eta_m = \cos\theta_m, \qquad \theta_m = \frac{2m-1}{2M}\pi \qquad (m = 1, 2, \dots, M).$$

Using Eq. (3.5) and the relations

$$\frac{1}{\pi}\int_{0}^{\pi} \frac{\cos M\tau}{\cos\tau - \cos\theta}\,d\tau = \frac{\sin M\theta}{\sin\theta}, \qquad 0 \leqslant \theta \leqslant \pi,$$

$$\int_{-1}^{1} \frac{F(\tau)}{\sqrt{1-\tau^2}}\,d\tau = \frac{\pi}{M}\sum_{k=1}^{M} F(\cos\theta_k),$$

we obtain the quadrature formulas

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{g_*(\tau)}{\tau-\eta}\,d\tau = \frac{1}{M\sin\theta}\sum_{k=1}^{M} g_k^0 \sum_{m=0}^{M-1}\cos m\theta_k\,\sin m\theta,$$

$$\frac{1}{2\pi}\int_{-1}^{1} g_*(\tau)B(\eta,\tau)\,d\tau = \frac{1}{2M}\sum_{k=1}^{M} g_k^0 B(\eta,\tau_k). \tag{3.6}$$

Using these formulas, we can reduce the integral equation (3.3) to a system of algebraic equations with respect to the approximate values g_k^0 of the sought function at the node points. In the problem considered, one end of the crack reaches the surface of the free hole. The stresses at this end of the crack are bounded. After certain transformations, the integral equation is replaced by a system of algebraic equations

$$\sum_{k=1}^{M} a_{m,k} g_k^0 + \frac{1}{2} H_*(\eta_m) = f_*(\eta_m), \qquad m = 1, 2, \ldots, M-1,$$
$$\sum_{k=1}^{M} (-1)^{k+M} g_k^0 \tan \frac{\theta_k}{2} = 0, \tag{3.7}$$

where

$$a_{m,k} = \frac{1}{2M}\Big(\frac{1}{\sin\theta_m} \cot \frac{\theta_m + (-1)^{|m-k|}\theta_k}{2} + B(\tau_m, \eta_k)\Big).$$

The right sides of system (3.7) contain the unknown values of the contact stresses $f_*(\eta_m)$ at the node points that belong to the tip contact zone. The condition that allows us to determine the unknown contact stresses arising on the crack faces in the tip contact zones is the absence of crack opening in these zones (the second condition on L''). In the problem considered, it is more convenient to write this additional condition for the derivative of opening of crack face displacements as

$$g(x) = \frac{2\mu}{1+\varkappa_0} \frac{\partial}{\partial x} [v^+(x,0) - v^-(x,0)] = 0, \tag{3.8}$$

where x is the affix of the points of the crack faces in the tip contact zone (l_1, l).

Requiring that conditions (3.8) are satisfied at the node points contained in the tip contact zone (l_1, l), we obtain the missing equations for determining the approximate values of the contact stresses $p(t_{m_1})$ at the node points:

$$g(t_{m_1}) = 0, \qquad m_1 = 1, 2, \ldots, M_1 \tag{3.9}$$

(M_1 is the number of node points that belong to the tip contact zone of the crack).

To close system (2.10), (2.11), (2.14), (3.7), (3.9), we need one more equation determining the tip zone size. The size of the tip contact zone is determined from the condition of finite stresses in the vicinity of the crack tip $x = \pm l$. Writing the condition of finite stresses at the point $x = \pm l$, we obtain the missing equation in the form

$$\sum_{m=1}^{M} (-1)^m g_k^0 \cot \frac{2m-1}{4M} \pi = 0. \tag{3.10}$$

As the size of the tip contact zone is unknown, the algebraic system (2.10), (2.11), (2.14), (3.7), (3.9), (3.10) is nonlinear. For a specified external load, the systems of equations with respect to α_{2k}, β_{2k}, g_k^0, F_{mn}, $p(t_{m_1})$, and l_1 allow us to determine the stress–strain state of the perforated isotropic medium reinforced with a regular system of stringers and having cracks with partly contacting faces, the contact stresses, and the size of the tip contact zone. To solve the algebraic systems (2.10), (2.11), (2.14), (3.7), (3.9), and (3.10), we use the method of consecutive approximations, which is based on the following procedure. For a certain value of l_{1*}, we solve system (2.10), (2.11), (2.14), (3.7), (3.9) with respect to the unknown α_{2k}, β_{2k}, $g_1^0, g_2^0, \ldots, g_M^0$, and $p_1, p_2, \ldots, p_{M_1}$ and $N_1 \times N_2$ unknown point forces. The values of l_{1*} and other quantities found are substituted into Eq. (3.10). Generally speaking, the chosen value of the parameter l_{1*}, the corresponding values of α_{2k}, β_{2k}, $g_1^0, g_2^0, \ldots, g_M^0$, and $p_1, p_2, \ldots, p_{M_1}$, and the point forces do not satisfy Eq. (3.10) of the system considered. Therefore, the calculations are repeated with different values of the parameter l_{1*} until Eq. (3.10) is satisfied with specified accuracy.

The calculations were performed with for the following geometric parameters of the reinforced plate: $\nu = 0.3$, $\varepsilon_1 = a_0/L = 0.01$, $\varepsilon = y_0/L = 0.15$ and 0.25, $E = 7.1 \cdot 10^4$ MPa (V95 alloy), $E_s = 11.5 \cdot 10^4$ MPa (aluminum–steel composite), and $A_s/(y_0 h) = 1$. The number of stringers and attachment points was assumed to be finite: 6, 10, or 14.

A parametric analysis of the contact stresses $p(x)$ as functions of the crack size and geometric parameters of the problem was performed.

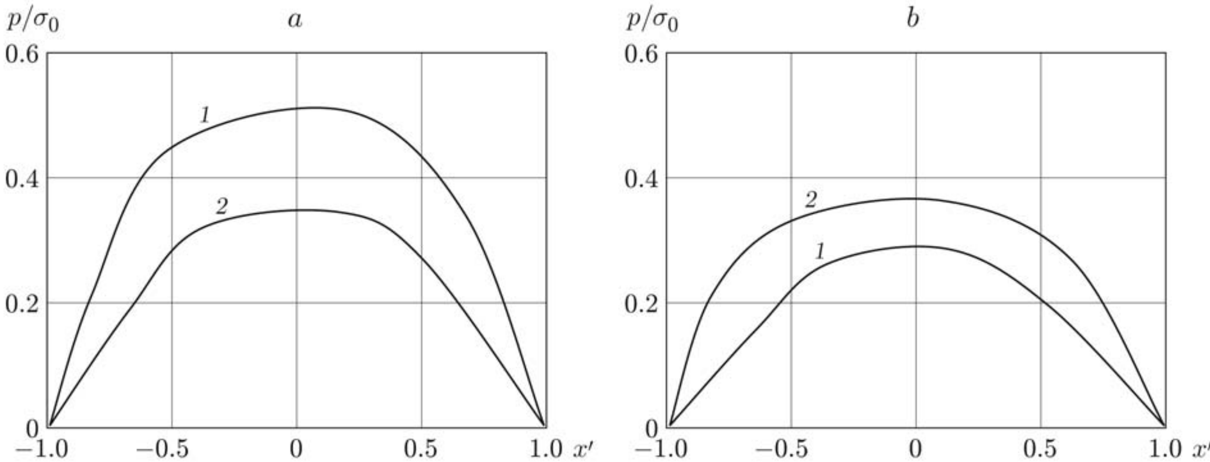

Fig. 2. Distributions of the contact stresses along the tip zone of the crack at $\varepsilon = 0.25$ and $l_* = l/L = 0.7$ (a) and 0.5 (b); curves 1 and 2 refer to $\lambda = 0.3$ and 0.5, respectively.

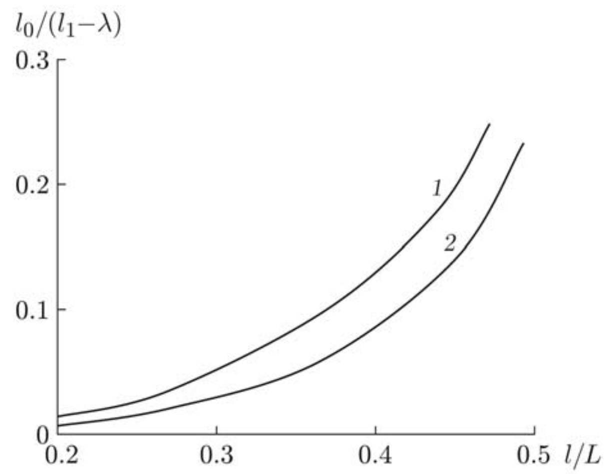

Fig. 3

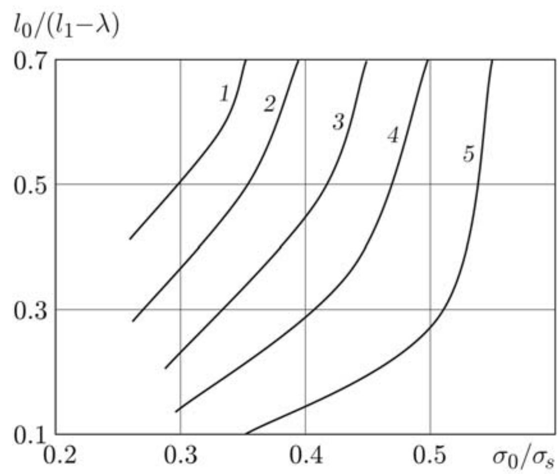

Fig. 4

Fig. 3. Quantity $l_0/(l_1 - \lambda)$ versus the dimensionless crack length l/L at $\varepsilon = 0.25$ and $l/\lambda = 1.5$ (1) and 1.25 (2).

Fig. 4. Quantity $l_0/(l_1 - \lambda)$ versus the dimensionless intensity of the tensile external load σ_0/σ_s for the hole radius $\lambda = 0.5$ (1), 0.4 (2), 0.3 (3), 0.2 (4), and 0.1 (5).

Figure 2 shows the calculated contact stresses p/σ_0 in the tip zone for different crack lengths. The dimensionless coordinates x' were used in the calculations:

$$x = \frac{l + l_1}{2} + \frac{l - l_1}{2}\, x'.$$

The greatest values of the contact stresses are reached in the middle of the contact zone where the crack faces approach each other.

Figure 3 shows the parameter $l_0/(l_1 - \lambda)$ as a function of the crack length l/L. The length of the tip contact zone $l_0/(l_1 - \lambda)$ versus the dimensionless intensity of the tensile external load σ_0/σ_s for different values of the hole radius λ is plotted in Fig. 4.

As is seen from the calculated results, the more frequently the attachment points are located, the greater the size of the tip contact zone of the cracks.

The analysis of the model of partial closure of cracks in a perforated isotropic medium reinforced by a regular system of transverse stringers is reduced to a parametric study of the algebraic systems (2.10), (2.11), (2.14), (3.7), (3.9), and (3.10) with different geometric and physical parameters of the problem. The relations obtained also allow one to solve the inverse problem, i.e., determine the positions of the stringers and their attachment points, as well as the stress state of the plate, which correspond to a specified contact zone of crack faces.

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